[10.1]

(1)

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{0} dx\ (-a)\ \cos\frac{n\pi x}{L} + \frac{1}{L}\int_{0}^{L} dx\ a\ \cos\frac{n\pi x}{L} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{0} dx\ (-a)\ \sin\frac{n\pi x}{L} + \frac{1}{L}\int_{0}^{L} dx\ a\ \sin\frac{n\pi x}{L} \\
&= \frac{2a}{L}\int_{0}^{L} dx\ \sin\frac{n\pi x}{L} \\
&= \frac{2a}{L}\left[-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L \\
&= \frac{2a}{n\pi}\{1-(-1)^n\} \\
&= \begin{cases} 0 & (n:\text{偶数}) \\ 2 & (n:\text{奇数}) \end{cases}
\end{aligned}
$$

よって

$$
f_{\mathrm{p}}(x) = \sum_{n=1}^{\infty}\frac{4a}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{L}
$$

(2)

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{b} dx\ a\frac{L+x}{L+b}\cos\frac{n\pi x}{L} + \frac{1}{L}\int_{b}^{L} dx\ a\frac{L-x}{L-b}\cos\frac{n\pi x}{L} \\
&= \frac{2aL^2}{(n\pi)^2(L^2-b^2)}\left\{\cos\frac{n\pi b}{L} - (-1)^n\right\} \\
b_n &= \frac{1}{L}\int_{-L}^{b} dx\ a\frac{L+x}{L+b}\sin\frac{n\pi x}{L} + \frac{1}{L}\int_{b}^{L} dx\ a\frac{L-x}{L-b}\sin\frac{n\pi x}{L} \\
&= \frac{2aL^2}{(n\pi)^2(L^2-b^2)}\ \sin\frac{n\pi b}{L}
\end{aligned}
$$

[10.2]

$$
\begin{aligned}
f_{\mathrm{p}}(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\frac{\mathrm{e}^{i\frac{n\pi x}{L}} + \mathrm{e}^{-i\frac{n\pi x}{L}}}{2} + b_n\frac{\mathrm{e}^{i\frac{n\pi x}{L}} - \mathrm{e}^{-i\frac{n\pi x}{L}}}{2i}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( \frac{a_n - i\,b_n}{2}\,\mathrm{e}^{i\frac{n\pi x}{L}} + \frac{a_n + i\,b_n}{2}\,\mathrm{e}^{-i\frac{n\pi x}{L}} \right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - i\,b_n}{2}\,\mathrm{e}^{i\frac{n\pi x}{L}} + \sum_{n=-\infty}^{-1} \frac{a_n + i\,b_n}{2}\,\mathrm{e}^{-i\frac{n\pi x}{L}} \\
&= \sum_{n=-\infty}^{\infty} \frac{a_n - i\,b_n}{2}\,\mathrm{e}^{i\frac{n\pi x}{L}}\,, \quad (a_{-n} = a_n\,,\, b_{-n} = -b_n\,,\, b_0 = 0)
\end{aligned}
$$

$$
c_n = \frac{a_n - i\,b_n}{2}
$$

$$
\begin{aligned}
c_n^* &= \frac{1}{2L}\int_{-\infty}^{\infty} dx\ f^*(x)\,\mathrm{e}^{ik_n x} \\
&= \frac{1}{2L}\int_{-\infty}^{\infty} dx\ f(x)\,\mathrm{e}^{-i(-k_n)x} \\
&= c_{-n} \qquad (-k_n = k_{-n})
\end{aligned}
$$

[10.3]

(1)

$$
\begin{aligned}
u_t &= \sum_{n=1}^{\infty} (-\omega_n\,f_n \sin\omega_n t + g_n \cos\omega_n t)\sin k_n x \\
u_t(x,0) &= \sum_{n=1}^{\infty} g_n \sin k_n x = g(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{\ell}\int_0^{\ell} dx\ g(x)\sin k_n x &= \frac{2}{\ell}\int_0^{\ell} dx\ \sum_{n'=1}^{\infty} g_{n'} \sin k_{n'} x \sin k_n x \\
&= \frac{2}{\ell}\sum_{n'=1}^{\infty} g_{n'} \int_0^{\ell} dx\ \sin k_{n'} x \sin k_n x \\
&= \frac{2}{\ell}\sum_{n'=1}^{\infty} g_{n'}\,\frac{\ell}{2}\,\delta_{n,n'} \\
&= g_n
\end{aligned}
$$

ここで

$$
\int_0^{\ell} dx\ \sin k_{n'} x \sin k_n x = \frac{\ell}{2}\,\delta_{n,n'}
$$

を用いた。一方、

$$
u(x,0) = \sum_{n=1}^{\infty} f_n \sin k_n x = f(x)
$$

$$
\begin{aligned}
\frac{2}{\ell}\int_0^\ell dx\ f(x)\sin k_n x &= \frac{2}{\ell}\sum_{n'=1}^{\infty} f_{n'}\int_0^\ell dx\ \sin k_{n'}x \sin k_n x \\
&= f_n
\end{aligned}
$$

$$
\begin{aligned}
u_{tt} &= \sum_{n=1}^{\infty}\left(-\omega_n^2\, f_n\cos\omega_n t - \omega_n\, g_n \sin\omega_n t\right)\sin k_n x \\
u_{xx} &= -\sum_{n=1}^{\infty}\left(f_n\cos\omega_n t + \frac{g_n}{\omega_n}\sin\omega_n t\right) k_n^2 \sin k_n x \\
u_{tt} - c^2\, u_{xx} &= \sum_{n=1}^{\infty}(-\omega_n^2 + c^2\, k_n^2)\left(f_n\cos\omega_n t + \frac{g_n}{\omega_n}\sin\omega_n t\right)\sin k_n x \\
&= 0 \qquad (\omega_n^2 = c^2\, k_n^2)
\end{aligned}
$$

(2)

$$
\begin{aligned}
E &= \frac{1}{2}\int_0^\ell dx\ \left(\rho\, u_t^2 + T\, u_x^2\right) \\
&= \frac{1}{2}\int_0^\ell dx\ \Bigg[\rho\sum_{n,n'}(-\omega_n\, f_n\sin\omega_n t + g_n\cos\omega_n t)\sin k_n x \\
&\qquad \times(-\omega_{n'}\, f_{n'}\sin\omega_{n'}t + g_{n'}\cos\omega_{n'}t)\sin k_{n'}x \\
&\qquad +T\sum_{n,n'}\left(f_n\cos\omega_n t + \frac{g_n}{\omega_n}\sin\omega_n t\right) k_n \sin k_n x \\
&\qquad \times\left(f_{n'}\cos\omega_{n'}t + \frac{g_{n'}}{\omega_{n'}}\sin\omega_{n'}t\right) k_{n'}\sin k_{n'}x\Bigg] \\
&= \frac{\ell}{4}\sum_n\left[\rho\,(-\omega_n\, f_n\sin\omega_n t + g_n\cos\omega_n t)^2 + T\, k_n^2\left(f_n\cos\omega_n t + \frac{g_n}{\omega_n}\sin\omega_n t\right)^2\right] \\
&= \frac{\ell\rho}{4}\sum_n \omega_n^2\left(f_n^2 + \frac{g_n^2}{\omega_n^2}\right)
\end{aligned}
$$

(3)

$$
\begin{aligned}
f_n &= \frac{2}{\ell}\int_0^\ell dx\ x\,(\ell - x)\sin k_n x \\
&= \frac{2}{\ell}\int_0^\ell dx\ x\,\ell\sin k_n x - \frac{2}{\ell}\int_0^\ell dx\ x^2\sin k_n x \\
&= \frac{4}{\ell k_n^3}\left\{1 - (-1)^n\right\}
\end{aligned}
$$

$$= \begin{cases} 0 & (n = \text{偶数}) \\ \dfrac{8}{\ell k_n^3} & (n = \text{奇数}) \end{cases}$$

よって

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{\ell k_{2n-1}^3} \cos \omega_{2n-1} t \, \sin k_{2n-1} x$$