[11.1]

(a)

$$\begin{aligned}
\mathcal{F}[a\,f(x)+b\,g(x)] &= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx}[af(x)+bg(x)] \\
&= a\int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx} f(x) + b\int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx} g(x) \\
&= a\,\hat{f}(k) + b\,\hat{g}(k)
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{F}[f(x+a)] &= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx} f(x+a) \\
&= \int_{-\infty}^{\infty} dx'\ \mathrm{e}^{-ik(x'-a)} f(x') \qquad (x+a=x') \\
&= \hat{f}(k)\ \mathrm{e}^{ika}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathcal{F}[\mathrm{e}^{iax}\,f(x)] &= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-i(k-a)x}\ f(x) \\
&= \hat{f}(k-a)
\end{aligned}$$

(d)

$$\begin{aligned}
\mathcal{F}[f^*(x)] &= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx}\ f^*(x) \\
&= \left[\int_{-\infty}^{\infty} dx\ \mathrm{e}^{ikx}\ f(x)\right]^* \\
&= \hat{f}^*(k)
\end{aligned}$$

(e)

$$\begin{aligned}
\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right] &= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx}\ \frac{d^n f(x)}{dx^n} \\
&= \left[\mathrm{e}^{-ikx}\ \frac{d^{n-1} f(x)}{dx^{n-1}}\right]_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx}\ \frac{d^{n-1} f(x)}{dx^{n-1}} \\
&= \cdots \\
&= (ik)^n \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx}\ f(x) \\
&= (ik)^n\,\hat{f}(k)
\end{aligned}$$

[11.2]

(1)

$$\begin{aligned}
\mathcal{F}[f*g](k) &= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx} \int_{-\infty}^{\infty} dx'\ f(x-x')\,g(x') \\
&= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ik(x-x')}\,f(x-x')\,\mathrm{e}^{-ikx'}\,g(x') \\
&= \int_{-\infty}^{\infty} dx''\ f(x'')\,\mathrm{e}^{-ikx''} \int_{-\infty}^{\infty} dx'\ g(x')\,\mathrm{e}^{-ikx'} \\
&= \hat{f}(k)\,\hat{g}(k)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}[fg](k) &= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx}\ f(x)\,g(x) \\
&= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx}\ f(x)\frac{1}{2\pi}\int_{-\infty}^{\infty} du\ \mathrm{e}^{iux}\,\hat{g}(u) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-i(k-u)x}\ f(x)\,\hat{g}(u) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} du\ \hat{f}(k-u)\,\hat{g}(u) \\
&= \frac{1}{2\pi}(\hat{f}*\hat{g})(k)
\end{aligned}$$

(2)

$$\begin{aligned}
(f*g)(x) &= \int_{-\infty}^{\infty} dx'\ f(x-x')\,g(x') \\
&= \int_{-\infty}^{\infty} dy\ f(y)\,g(x-y) \qquad (x-x'=y) \\
&= (g*f)(x)
\end{aligned}$$

$$\begin{aligned}
(f*g)*h &= \int_{-\infty}^{\infty} dx'\ (f*g)(x-x')\,h(x') \\
&= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx''\ f(x-x'-x'')\,g(x'')\,h(x') \\
&= \int_{-\infty}^{\infty} dy'' \int_{-\infty}^{\infty} dy'\ f(x-y')\,g(y'-y'')\,h(y'') \qquad (x'=y'', x''=y'-y'') \\
&= \int_{-\infty}^{\infty} dy'\ f(x-y') \int_{-\infty}^{\infty} dy''\ g(y'-y'')\,h(y'') \\
&= f*(g*h)(x)
\end{aligned}$$

(3)

$$
\begin{aligned}
(f*g*h)(x) &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx''\ f(x-x')\, g(x'-x'')\, h(x'') \\
&= \int_{-\infty}^{\infty} dx'\ f(x-x') \int_{-\infty}^{\infty} dx''\ , g(x'-x'')\, h(x'') \\
&= \int_{-\infty}^{\infty} dx'\ f(x-x')\, G(x')\,, \qquad G(x') \equiv \int_{-\infty}^{\infty} dx''\ , g(x'-x'')\, h(x'') \\
&= (f*G)(x)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}[f*G](k) &= \hat{f}(k)\, \hat{G}(k) \\
\hat{G}(k) &= \hat{g}(k)\, \hat{h}(k)
\end{aligned}
$$

よって

$$
\mathcal{F}[f*g*h](k) = \hat{f}(k)\, \hat{g}(k)\, \hat{h}(k)
$$

[11.3]

(1)

$$
\begin{aligned}
\hat{\psi}(k) &= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx} \int_{-\infty}^{\infty} dx'\ K(x,x') \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\ \mathrm{e}^{ik'x'}\, \hat{\varphi}(k') \\
&= \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\ \mathrm{e}^{-ikx+ik'x'}\ K(x,x')\, \hat{\varphi}(k') \\
&= \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\ \hat{K}(k,-k')\, \hat{\varphi}(k')
\end{aligned}
$$

(2)

$$
\begin{aligned}
\hat{\psi}(k) &= \int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx} \int_{-\infty}^{\infty} dx'\ G(x-x') \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\ \mathrm{e}^{ik'x'}\, \hat{\varphi}(k') \\
&= \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\ \mathrm{e}^{-ikx+ik'x'}\ G(x-x')\, \hat{\varphi}(k') \\
&= \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx'\ \mathrm{e}^{-iky-ikx'}\mathrm{e}^{ik'x'}\ G(y)\, \hat{\varphi}(k') \\
&= \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} dy\ \mathrm{e}^{-iky}\ G(y) \int_{-\infty}^{\infty} dx'\ \mathrm{e}^{-i(k-k')x'}\ \hat{\varphi}(k') \\
&= \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \hat{G}(k) 2\pi \delta(k-k')\, \hat{\varphi}(k') \\
&= \hat{G}(k) \hat{\varphi}(k)
\end{aligned}
$$

[11.4]

(1)

$$\hat{f}(k) = \int_0^\infty dx\, \mathrm{e}^{-ikx}\, \mathrm{e}^{-ax}$$

$$= \frac{1}{ik+a}$$

(2)

$$\hat{f}(k) = \int_{-\infty}^\infty dx\, \mathrm{e}^{-ikx}\, \mathrm{e}^{-a|x|}$$

$$= \int_0^\infty dx\, \mathrm{e}^{-ikx}\, \mathrm{e}^{-ax} + \int_0^\infty dx\, \mathrm{e}^{ikx}\, \mathrm{e}^{-ax}$$

$$= \frac{1}{ik+a} + \frac{1}{-ik+a}$$

(3)

$$\hat{f}(k) = \int_{-\infty}^\infty dx\, \mathrm{e}^{-ikx}\, \mathrm{e}^{-ax^2}$$

$$= \int_{-\infty}^\infty dx\, \mathrm{e}^{-a\left(x+\frac{ik}{2a}\right)^2-\frac{k^2}{4a}}$$

$$= \sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\frac{k^2}{4a}}$$

(4)

$$\hat{f}(k) = \int_{-\infty}^\infty dx\, \mathrm{e}^{-ikx}\, \frac{1}{x^2+a^2}$$

x=-ia に一位の極（下半面を周る経路をとる）

$$= -2\pi i \frac{\mathrm{e}^{-ka}}{-2ia}$$

$$= \frac{\pi}{a}\,\mathrm{e}^{-ka}$$

[11.5]

(1)　両辺を Fourier 変換

$$c_0\,(ik)^n\,\hat{f}(k) + c_1\,(ik)^{n-1}\,\hat{f}(k) + \cdots + c_{n-1}\,(ik)\,\hat{f}(k) + c_n\,\hat{f}(k) = \hat{F}(k)$$

$$= \sum_{m=0}^{n} c_m\,(ik)^{n-m}\,\hat{f}(k)$$

よって

$$\hat{f}(k) = \left[\sum_{m=0}^{n} c_m\,(ik)^{n-m}\right]^{-1} \hat{F}(k)$$

(2)

$$\int_{-\infty}^{\infty} dx\ \mathrm{e}^{-ikx}\,\delta(x) = 1$$

より

$$\sum_{m=0}^{n} c_m\,(ik)^{n-m}\,\hat{G}(k) = 1$$

$$\begin{aligned}
\hat{G}(k) &= \left[\sum_{m=0}^{n} c_m\,(ik)^{n-m}\right]^{-1} \\
\hat{f}(k) &= \hat{G}(k)\hat{F}(k)
\end{aligned}$$

たたみ込みから

$$f(x) = \int_{-\infty}^{\infty} dx'\ G(x-x')\,F(x')$$

(3)

$F(x)$=$a\,\mathrm{e}^{i(kx+\varphi)}$ の時

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} dx'\ \frac{1}{2\pi}\int_{-\infty}^{\infty} dk'\ \mathrm{e}^{ik'(x-x')}\,\hat{G}(k')\,a\,\mathrm{e}^{i(kx+\varphi)} \\
&= \frac{a}{2\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dk'\ \mathrm{e}^{i(k-k')x'}\,\hat{G}(k')\,\mathrm{e}^{ik'x}\mathrm{e}^{i\varphi} \\
&= a\,\hat{G}(k)\,\mathrm{e}^{i(kx+\varphi)}
\end{aligned}$$

$F(x)$=$a\ \cos(kx+\varphi)$ の時

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} dx'\ \frac{1}{2\pi}\int_{-\infty}^{\infty} dk'\ \mathrm{e}^{ik'(x-x')}\,\hat{G}(k')\,a\ \cos(kx+\varphi) \\
&= \frac{a}{2}\int_{-\infty}^{\infty} dx'\ \frac{1}{2\pi}\int_{-\infty}^{\infty} dk'\ \mathrm{e}^{ik'(x-x')}\,\hat{G}(k')\,\left\{\mathrm{e}^{i(kx+\varphi)} + \mathrm{e}^{-i(kx+\varphi)}\right\} \\
&= \frac{a}{2}\,\hat{G}(k)\,\mathrm{e}^{i(kx+\varphi)} + \frac{a}{2}\,\hat{G}(-k)\,\mathrm{e}^{-i(kx+\varphi)} \\
&= a\,\mathrm{Re}\left[\hat{G}(k)\,\mathrm{e}^{i(kx+\varphi)}\right] \quad \left(\hat{G}(-k) = \hat{G}^*(k)\right)
\end{aligned}$$

[11.6]

(1)

$$m(i\omega)^2\hat{x}(\omega) + \gamma(i\omega)\hat{x}(\omega) + \omega_0^2\hat{x}(\omega) = \hat{F}(\omega)$$

$$\hat{x}(\omega) = \frac{\hat{F}(\omega)}{-m\omega^2 + i\gamma\omega + \omega_0^2}$$

Green 関数を考える

$$\left(m\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right)\,G(t) = \delta(t)$$

$$
\begin{aligned}
\hat{G}(\omega) &= \frac{1}{-m\omega^2 + i\gamma\omega + \omega_0^2} \\
\hat{x}(\omega) &= \hat{G}(\omega)\,\hat{F}(\omega) \\
x(t) &= \int_{-\infty}^{\infty} dt'\, G(t-t')\, F(t')
\end{aligned}
$$

(3)

$F(t) = F_0 \cos\omega t$

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} dt'\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, \mathrm{e}^{i\omega'(t-t')}\, \hat{G}(\omega')\, F_0 \cos\omega t' \\
&= \frac{F_0}{2} \int_{-\infty}^{\infty} dt'\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, \mathrm{e}^{i\omega'(t-t')}\, \hat{G}(\omega')\, \left\{ \mathrm{e}^{i\omega t'} + \mathrm{e}^{-i\omega t'} \right\} \\
&= \frac{F_0}{4\pi} \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} dt'\, \mathrm{e}^{i(\omega'-\omega)t'}\, \hat{G}(\omega')\, \mathrm{e}^{i\omega' t} \\
&\quad + \frac{F_0}{4\pi} \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} dt'\, \mathrm{e}^{i(\omega'+\omega)t'}\, \hat{G}(\omega')\, \mathrm{e}^{i\omega' t} \\
&= \frac{F_0}{2}\, \hat{G}(\omega)\, \mathrm{e}^{i\omega t} + \frac{F_0}{2}\, \hat{G}(-\omega)\, \mathrm{e}^{-i\omega t} \\
&= F_0\, \mathrm{Re}\left[\hat{G}(\omega)\, \mathrm{e}^{i\omega t} \right]
\end{aligned}
$$