[12.1]

(1) (a)

$$\begin{aligned}\mathcal{L}[bf(x)+cg(x)] &= \int_0^\infty dx\ \mathrm{e}^{-sx}[bf(x)+cg(x)] \\ &= b\int_0^\infty dx\ \mathrm{e}^{-sx}f(x) + c\int_0^\infty dx\ \mathrm{e}^{-sx}g(x) \\ &= b\tilde{f}(s)+c\tilde{g}(s)\end{aligned}$$

(b)

$$\begin{aligned}\mathcal{L}[\mathrm{e}^{cx}f(x)] &= \int_0^\infty dx\ \mathrm{e}^{-sx}\mathrm{e}^{cx}f(x) \\ &= \int_0^\infty dx\ \mathrm{e}^{-(s-c)x}f(x) \\ &= \tilde{f}(s-c)\end{aligned}$$

(c)

$$\begin{aligned}\mathcal{L}[f(ax)] &= \int_0^\infty dx\ \mathrm{e}^{-sx}f(ax) \\ &= \int_0^\infty \frac{dx'}{a}\ \mathrm{e}^{-sx'/a}f(x') \quad ax \equiv x' \\ &= \frac{1}{a}\tilde{f}\left(\frac{s}{a}\right)\end{aligned}$$

(d)

$$\begin{aligned}\mathcal{L}[f(x-a)\theta(x-a)] &= \int_0^\infty dx\ \mathrm{e}^{-sx}f(x-a)\theta(x-a) \\ &= \int_{-a}^\infty dx'\ \mathrm{e}^{-s(x'+a)}f(x')\theta(x') \quad x-a \equiv x' \\ &= \int_0^\infty dx'\ \mathrm{e}^{-sx'}f(x')\cdot \mathrm{e}^{-sa} \\ &= \mathrm{e}^{-sa}\tilde{f}(s)\end{aligned}$$

(e)

$$\begin{aligned}\mathcal{L}[x^n] &= \int_0^\infty dx\ \mathrm{e}^{-sx}x^n \\ &= \left[-\frac{1}{s}\mathrm{e}^{-sx}x^n\right]_0^\infty + \frac{1}{s}\int_0^\infty dx\ \mathrm{e}^{-sx}nx^{n-1} \\ &= \frac{n}{s}\left[-\frac{1}{s}\mathrm{e}^{-sx}x^{n-1}\right]_0^\infty + \frac{n(n-1)}{s^2}\int_0^\infty dx\ \mathrm{e}^{-sx}nx^{n-2}\end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= \frac{n!}{s^n}\int_0^\infty dx\ \mathrm{e}^{-sx} \\
&= \frac{n!}{s^n}\left[-\frac{1}{s}\mathrm{e}^{-sx}\right]_0^\infty \\
&= \frac{n!}{s^{n+1}}
\end{aligned}
$$

(f)

$$
\begin{aligned}
\mathcal{L}[x^n f(x)] &= \int_0^\infty dx\ \mathrm{e}^{-sx}x^n f(x) \\
&= \int_0^\infty dx\ (-1)^n \frac{d^n}{ds^n}\mathrm{e}^{-sx}f(x) \\
&= (-1)^n \frac{d^n}{ds^n}\int_0^\infty dx\ \mathrm{e}^{-sx}f(x) \\
&= (-1)^n \frac{d^n}{ds^n}\tilde{f}(s)
\end{aligned}
$$

(g)

$$
\mathcal{L}\left[\frac{d^n f(x)}{dx^n}\right] = \int_0^\infty dx\ \mathrm{e}^{-sx}\frac{d^n f(x)}{dx^n}
$$

n=1 の時

$$
\begin{aligned}
\mathcal{L}\left[\frac{df(x)}{dx}\right] &= \int_0^\infty dx\ \mathrm{e}^{-sx}\frac{df(x)}{dx} \\
&= \left[\mathrm{e}^{-sx}f(x)\right]_0^\infty + \int_0^\infty dx\ s\mathrm{e}^{-sx}f(x) \\
&= -f(0) + s\tilde{f}(s)
\end{aligned}
$$

n=k の時、**Note 12b**(g) が成り立つと仮定する。n=k+1 の時

$$
\begin{aligned}
\mathcal{L}\left[\frac{d^{k+1} f(x)}{dx^{k+1}}\right] &= \int_0^\infty dx\ \mathrm{e}^{-sx}\frac{d^{k+1} f(x)}{dx^{k+1}} \\
&= \left[\mathrm{e}^{-sx}\frac{d^k}{dx^k}f(x)\right]_0^\infty + s\int_0^\infty dx\ \mathrm{e}^{-sx}\frac{d^k}{dx^k}f(x) \\
&= -\frac{d^k}{dx^k}f(0) + s\mathcal{L}\left[\frac{d^k f(x)}{dx^k}\right] \\
&= -\frac{d^k}{dx^k}f(0) + s\left\{s^k\tilde{f}(s) - \sum_{m=0}^{k-1} s^{k-m-1}\frac{d^m}{dx^m}f(0)\right\} \\
&= s^{k+1}\tilde{f}(s) - \sum_{m=0}^{k} s^{(k+1)-m-1}\frac{d^m}{dx^m}f(0)
\end{aligned}
$$

よって n=k+1 の時も成り立つ。

(2) (a)

$$\mathcal{L}\left[\int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n\ f(x_n)\right] = \int_0^\infty dx\ \mathrm{e}^{-sx} \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n\ f(x_n)$$

n 回部分積分を行うと

$$= \left[-\frac{1}{s}\mathrm{e}^{-sx} \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n\ f(x_n)\right]_0^\infty + \frac{1}{s}\int_0^\infty dx\ \mathrm{e}^{-sx} \int_0^x dx_2 \cdots \int_0^{x_{n-1}} dx_n\ f(x_n)$$

$$= \cdots$$

$$= \frac{1}{s^n}\int_0^\infty dx\ \mathrm{e}^{-sx} f(x)$$

$$= \frac{1}{s^n}\tilde{f}(s)$$

(b)

$$\int_s^\infty ds_1 \int_{s_1}^\infty ds_2 \cdots \int_{s_{n-1}}^\infty ds_n \int_0^\infty dx\ \mathrm{e}^{-s_n x} f(x)$$

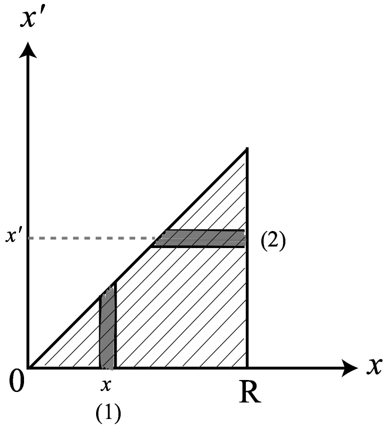
$$= \int_s^\infty ds_1 \int_{s_1}^\infty ds_2 \cdots \int_{s_{n-2}}^\infty ds_{n-1} \int_0^\infty dx\ \left[-\frac{1}{x}\mathrm{e}^{-s_n x}\right]_{s_{n-1}}^\infty f(x)$$

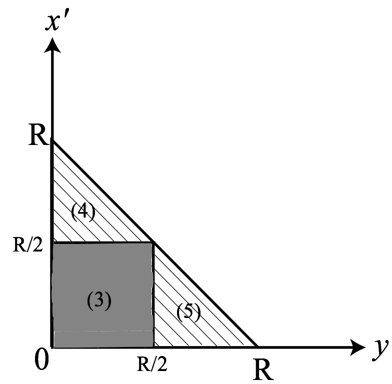
$$= \int_s^\infty ds_1 \int_{s_1}^\infty ds_2 \cdots \int_{s_{n-2}}^\infty ds_{n-1} \int_0^\infty dx\ \frac{1}{x}\mathrm{e}^{-s_{n-1} x} f(x)$$

$$= \cdots$$

$$= \int_0^\infty dx\ \frac{1}{x^n}\mathrm{e}^{-sx} f(x)$$

[12.2]





$$\int_0^R dx\,\mathrm{e}^{-sx}\int_0^x dx'\,f(x-x')g(x') \quad \cdots \quad (1)$$

$$= \int_0^R dx'\,\mathrm{e}^{-sx'}g(x')\int_{x'}^R dx\,\mathrm{e}^{-s(x-x')}f(x-x') \quad \cdots \quad (2)$$

x-x' を y とおく

$$= \int_0^{R/2} dx'\mathrm{e}^{-sx'}g(x')\int_0^{R/2} dy\mathrm{e}^{-sy}f(y) \quad \cdots \quad (3)$$

$$+ \int_{R/2}^R dx'\mathrm{e}^{-sx'}g(x')\int_0^{R-x'} dy\mathrm{e}^{-sy}f(y) \quad \cdots \quad (4)$$

$$+ \int_{R/2}^R dy\mathrm{e}^{-sy}f(y)\int_0^{R-y} dx'\mathrm{e}^{-sx'}g(x') \quad \cdots \quad (5)$$

(4) は

$$|(4)| \leq \int_{R/2}^R dx'\mathrm{e}^{-sx'}|g(x')| \cdot \int_0^{R-x'} dy\mathrm{e}^{-sy}|f(y)| \to 0 \quad (R \to \infty) \tag{1}$$

(5) についても同様。$R \to \infty$ で第一項は

$$\int_0^\infty dx'\mathrm{e}^{-sx'}g(x')\int_0^\infty dy\mathrm{e}^{-sy}f(y) = \tilde{g}(s)\tilde{f}(s) \quad \text{(証明終)}$$

逆変換

$$\begin{aligned}\tilde{f}(s)\tilde{g}(s) &= \int_0^\infty dx_1\,\mathrm{e}^{-sx_1}f(x_1)\int_0^\infty dx_2\,\mathrm{e}^{-sx_2}g(x_2) \\ &= \int_0^\infty dx_1 int_0^\infty dx_2\,\mathrm{e}^{-s(x_1+x_2)}f(x_1)g(x_2)\end{aligned}$$

x_1+x_2=x、x_2=x' とおくと、

$$0 \leq x_1 \leq \infty\,,\; 0 \leq x_2 \leq \infty \quad \to \quad 0 \leq x - x' \leq \infty\,,\; 0 \leq x' \leq \infty$$

$$\rightarrow \quad 0 \le x' \le x \; , \; 0 \le x \le \infty$$

よって

$$= \int_0^\infty dx \int_0^x dx' \; \mathrm{e}^{-sx} f(x-x') g(x')$$

[12.3]

(1)

$$\begin{aligned}
\tilde{f}(s) &= \int_0^\infty dx \; \mathrm{e}^{-sx}(\cos ax + i \sin ax) \\
&= \int_0^\infty dx \; \mathrm{e}^{-sx}\mathrm{e}^{iax} \\
&= \frac{1}{-s+ia}\mathrm{e}^{(-s+ia)x}\Big|_0^\infty \\
&= \frac{1}{s-ia} \\
&= \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}
\end{aligned}$$

よって

$$(a) \quad \frac{s}{s^2+a^2} \qquad\qquad (b) \quad \frac{a}{s^2+a^2}$$

(c)

$$\begin{aligned}
\tilde{f}(s) &= \int_0^\infty dx \; \mathrm{e}^{-sx}\frac{\mathrm{e}^{ax}+\mathrm{e}^{-ax}}{2} \\
&= \frac{1}{2}\left[\frac{1}{-s+a}\mathrm{e}^{-(s-a)x} + \frac{1}{-s-a}\mathrm{e}^{-(s+a)x}\right]_0^\infty \\
&= \frac{s}{s^2+a^2} \quad (\mathrm{Re}\, s > a)
\end{aligned}$$

(d) 同様にして

$$\tilde{f}(s) = \frac{a}{s^2-a^2} \quad (\mathrm{Re}\, s > a)$$

(2) (a)

$$\begin{aligned}
f(x) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds \; \mathrm{e}^{sx}\frac{s}{s^2+a^2} \qquad (s = \pm ia \text{ に一位の極}) \\
&= \frac{1}{2\pi i} \times 2\pi i \left\{\mathrm{e}^{iax}\frac{ia}{2ia} + \mathrm{e}^{-iax}\frac{-ia}{-2ia}\right\} \\
&= \cos ax
\end{aligned}$$

(b) 同様にして

$$f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds \; \mathrm{e}^{sx}\frac{a}{s^2+a^2}$$

$$
\begin{aligned}
&= e^{iax}\frac{a}{2ia} + e^{-iax}\frac{a}{-2ia} \\
&= \sin ax
\end{aligned}
$$

(c)

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\, e^{sx}\frac{s}{s^2-a^2} \qquad (s=\pm a \text{ に一位の極}) \\
&= e^{ax}\frac{a}{2a} + e^{-ax}\frac{-a}{-2a} \\
&= \cosh ax
\end{aligned}
$$

(d)

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\, e^{sx}\frac{a}{s^2-a^2} \qquad (s=\pm a \text{ に一位の極}) \\
&= e^{ax}\frac{a}{2a} + e^{-ax}\frac{a}{-2a} \\
&= \sinh ax
\end{aligned}
$$

[12.4]

(1)

$$
\begin{aligned}
\tilde{x}(s) &= \int_0^\infty dt\; x(t)e^{-st} \\
\widetilde{\frac{dx(s)}{dt}} &= -x(0) + s\tilde{x}(s)
\end{aligned}
$$

$$
\dot{x} + cx = F(t) \qquad \Rightarrow \qquad -x(0) + s\tilde{x}(s) + c\tilde{x}(s) = \tilde{F}(s)
$$

$$
\tilde{x}(s) = \frac{x_0 + \tilde{F}(s)}{s+c}
$$

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{x_0}{s+c}\right] &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\, e^{st}\frac{x_0}{s+c} \\
&= x_0 e^{-ct}
\end{aligned}
$$

$$
\mathcal{L}^{-1}\left[\frac{\tilde{F}(s)}{s+c}\right] = \int_0^t dz\; F(t-z)e^{-cz}
$$

よって

$$
x(t) = x_0 e^{-ct} + \int_0^t dz\; F(t-z)e^{-cz}
$$

(2)

$$\frac{\widetilde{d^2x(s)}}{dt^2} = s^2\tilde{x}(s) - sx(0) - \frac{d}{dt}x(0)$$

$$\ddot{x} + 2b\dot{x} + cx = F(t) \quad \Rightarrow \quad s^2\tilde{x}(s) - sx_0 - v_0 + 2bs\tilde{x}(s) - 2bx_0 + c\tilde{x}(s) = \tilde{F}(s)$$

$$\tilde{x}(s) = \frac{sx_0 + v_0 + 2bx_0 + \tilde{F}(s)}{s^2 + 2bs + c}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2+2bs+c}\right] &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\ \mathrm{e}^{st}\frac{1}{s^2+2bs+c} \\
&= \frac{\mathrm{e}^{[-b+(b^2-c)^{1/2}]t} - \mathrm{e}^{[-b-(b^2-c)^{1/2}]t}}{2(b^2-c)^{1/2}}
\end{aligned}$$

[12.5]

(1)

$$\begin{aligned}
\tilde{f}(s) &= \tilde{g}(s) + \tilde{h}(s)\tilde{f}(s) \qquad (\text{たたみ込みの Laplace 変換より}) \\
&\Rightarrow \quad \tilde{f}(s) = \frac{\tilde{g}(s)}{1-\tilde{h}(s)}
\end{aligned}$$

(2)

$$g(x) = x^2 \quad \Rightarrow \quad \tilde{g}(s) = \frac{2}{s^3} \quad , \quad h(x-x') = \sin(x-x') \quad \Rightarrow \quad \tilde{h}(s) = \frac{1}{s^2+1}$$

$$\begin{aligned}
\tilde{f}(s) &= 2\frac{s^2+1}{s^6} \\
f(x) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\ \mathrm{e}^{sx}2\frac{s^2+1}{s^6} \qquad (s=0\ \text{に六位の極}) \\
&= \frac{1}{60}(x^5 + 20x^3)
\end{aligned}$$

[12.6]

(1)

$$\begin{aligned}
\mathcal{L}[x^{\alpha-1}] &= \int_0^\infty dx\ \mathrm{e}^{-sx}x^{\alpha-1} \\
&= \int_0^\infty \frac{du}{s}\ \mathrm{e}^{-u}\left(\frac{u}{s}\right)^{\alpha-1} \quad (sx = u) \\
&= \frac{1}{s^\alpha}\int_0^\infty du\ u^{\alpha-1}\mathrm{e}^{-u} \\
&= \frac{1}{s^\alpha}\Gamma(\alpha)
\end{aligned}$$

(2)

$$f(u) = u^{\alpha-1} \quad , \quad g(1-u) = (1-u)^{\beta-1}$$

$$\mathcal{L}[f*g](s) = \tilde{f}(s)\tilde{g}(s) = \frac{\Gamma(\alpha)\Gamma(\beta)}{s^{\alpha+\beta}}$$

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+\beta}}\right] &= \left.\frac{x^{\alpha+\beta-1}}{(\alpha+\beta-1)!}\right|_{x=1} \qquad (\textbf{Note 12b}\ \text{(c)}) \\ &= \frac{1}{\Gamma(\alpha+\beta)} \end{aligned}$$

よって

$$(f*g)(u) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$